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On a characterization of some Newton-like methods of *R*-order at least three

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Abstract

In this paper, we extend to Banach spaces the result given by Gander in (Amer. Math. Monthly 92 (1985) 131) to obtain a characterization of Newton-like iterative process with *R*-order of convergence at least three. To do this, we consider the construction and the study of the semilocal convergence of a multiparametric family of iterative processes in Banach spaces for solving the nonlinear equation $F(x)=0$.

MSC: 45G10; 47H17; 65J15

Keywords: Iterative processes; *R*-order of convergence; Semilocal convergence

1. Introduction

The problem of solving the nonlinear scalar equation

$$f(t)=0, \tag{1}$$

where $f:(a,b)\subseteq \mathbb{R}\to\mathbb{R}$, has interested mathematicians for many centuries. For example, to calculate square roots, we have the famous Heron formula (75 b. C approx.): $t_{n+1}=\frac{1}{2}(t_n+R/t_n)$, where $f(t)=t^2-R$. This algorithm [9] was known by the Mesopotamica civilization two thousand years before Christ. Heron also obtained formulas to calculate roots of high orders.

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In this paper, we consider

$$F(x)=0, \tag{2}$$

where F is an operator $F: \Omega \subseteq X \to Y$ defined between two Banach spaces, so that we can study different problems: integral equations, boundary value problems, partial differential equations, etc. We approximate the solution of (2) by iterative processes. In this work, we study iterative processes with R-order of convergence at least three. Our goal is to try to obtain a theory, the most general possible, relative to these iterative processes. For that, firstly we consider

$$x_{n+1}=G(x_n). \tag{3}$$

To find (3), in scalar case (1), we notice the result obtained by Gander [5], where he studies methods of Newton type given by the following iteration function:

$$G(t)=t-H(L_f(t))\frac{f(t)}{f'(t)}, \tag{4}$$

where $L_f(t)$ is known as the "degree of logarithmic convexity" [7], that in the scalar case it is given by the following expression $L_f(t)=f(t)f''(t)/f'(t)^2$.

Gander (see for example [5]) characterizes the form that the iterative processes has with R-order of convergence at least three: *Let t^* be a simple zero of f and H a function such that $H(0)=1$, $H'(0)=\frac{1}{2}$ and $|H''(t)|<\infty$. The iteration $t_{n+1}=G(t_n)$, with $G(t)$ given by* (4), *has R-order of convergence at least three*.

The most known iterative processes with R-order of convergence at least three satisfies this result and they are given by iteration function (4) with

- Chebyshev's method [4]: $H(L_f(t_n))=1+\frac{1}{2}L_f(t_n)$,
- the super-Halley method [1]: $H(L_f(t_n))=1+\frac{1}{2}L_f(t_n)+\sum_{k\geqslant 2}\frac{1}{2}L_f(t_n)^k$,
- Halley's method [7]: $H(L_f(t_n))=1+\frac{1}{2}L_f(t_n)+\sum_{k\geqslant 2}\frac{1}{2^k}L_f(t_n)^k$,
- Ostrowski's method [5]: $H(L_f(t_n))=1+\frac{1}{2}L_f(t_n)+\sum_{k\geqslant 2}(-1)^k\binom{-1/2}{k}L_f(t_n)^k$,
- Euler's method [5]: $H(L_f(t_n))=1+\frac{1}{2}L_F(t_n)+\sum_{k\geqslant 2}(-1)^k 2^{k+1}\binom{1/2}{k+1}L_F(t_n)^k$.

Starting from the mentioned methods, it is clear that we can generalize these and obtain an iteration function for (3) by observing the sequential development of powers of the functions H that have these methods.

In the scalar case, we can consider

$$\begin{aligned} t_{n+1}&=G(t_n)=t_n-H(L_f(t_n))\frac{f(t)}{f'(t)},\\ H(L_f(t_n))&=1+\frac{1}{2}L_f(t_n)+\sum_{k\geqslant 2}A_kL_f(t_n)^k, \quad \{A_k\}_{k\geqslant 2}\subset \mathbb{R}^+, \end{aligned} \tag{5}$$

where $\{A_k\}_{k\geqslant 0}$ is a real decreasing sequence with $\sum_{k\geqslant 2}A_kt^k<+\infty$ for $|t|<r$, for what requires that $|L_f(t)|<r$ for the well definition of H.

First of all, we generalize (5) to Banach spaces.

2. Construction of new Newton-type iterations

To extend the above-mentioned iterations to Banach spaces, it is necessary to extend the degree of logarithmic convexity to operators defined in Banach spaces (see [7]).

Definition 1. Let F be a nonlinear twice Fréchet-differentiable operator in an open convex non-empty subset Ω of a Banach space X in another Banach space Y. If $x_0 \in \Omega$ and $\Gamma_0 = [F'(x_0)]^{-1} \in \mathscr{L}(Y, \Omega)$ exists, it is defined the "degree of logarithmic convexity'' operator as $L_F : \Omega \to \mathscr{L}(\Omega, \Omega)$, where for a given $x_0 \in \Omega$, it corresponds the linear operator $L_F(x_0) : \Omega \to \Omega$ such that

$$L_F(x_0)(x) = [F'(x_0)]^{-1} F''(x_0)[F'(x_0)]^{-1} F(x_0)(x), \quad x \in \Omega.$$

Our goal is the generalization of multiparametric family of iterative processes (5) to Banach spaces for solving Eq. (2).

Initially, we consider the family of iterative processes in the following form:

$$\begin{aligned} &x_{n+1} = G(x_n) = x_n - H(L_F(x_n))\Gamma_n F(x_n), \\ &H(L_F(x_n)) = I + \frac{1}{2} L_F(x_n) + \sum_{k \geqslant 2} A_k L_F(x_n)^k, \quad \{A_k\}_{k\geqslant 2} \subset \mathbb{R}^+, \end{aligned} \tag{6}$$

where $\{A_k\}_{k\geqslant 2}$ is a non-increasing sequence where $\sum_{k\geqslant 2} A_k t^k < +\infty$ for $|t| < r$.

If $L_F(x_n)$ exists and $\|L_F(x_n)\| < r$, $n \geqslant 0$, then (6) is well defined, see [3].

It is clear that this family of iterative processes, in the scalar case, has convergence at least three, since $H(0) = 1$, $H'(0) = \frac{1}{2}$ and $|H''(0)| < \infty$.

To observe that this family of iterative processes is well defined in Banach spaces, when the operator H is. If it is denoted $I = L_F(x)^0$, one has that the operator H is,

$$H(L_F(\_)) : \Omega \xrightarrow{L_F} \mathscr{L}(\Omega, \Omega) \xrightarrow{H} \mathscr{L}(\Omega, \Omega),$$

where it is associated to each x_n a "polynomial'' in $L_F(x_n)$, that is, $H(L_F(x_n)) = \sum_{k\geqslant 0} A_k L_F(x_n)^k$, with $A_0 = 1$ and $A_1 = \frac{1}{2}$. Besides, we denote $L_F(x_n)^k$ the composition $L_F(x)^k = \overbrace{L_F(x) \circ \cdots \circ L_F(x)}^{k}$ that is linear operator in Ω.

3. Semilocal convergence

In this section, we obtain a result of semilocal convergence, for the parametric family of iterative processes given in (6). For it, we use conditions of Kantorovich type for the operator F [8]. This type of conditions habitually allows the use of mayorizing sequences. However, in this case, it is not simple to bound the R-order of convergence of the iterative processes considered. For this, we introduce another technique based on the construction of recurrence relations, so that we can bound the R-order of convergence.

We consider Eq. (2), where F is a nonlinear operator under the previous conditions. Family of iterative processes (6) can be defined in the following way:

$$\begin{aligned}
&y_n = x_n - \Gamma_n F(x_n),\\
&x_{n+1} = G(x_n) = y_n + \tfrac{1}{2} L_F(x_n)\tilde{H}(L_F(x_n))(y_n - x_n),\\
&\tilde{H}(L_F(x_n)) = I + \sum_{k\geqslant 2} 2A_k L_F(x_n)^{k-1}, \quad \text{with } A_k \in \mathbb{R}^+,\ k\geqslant 2.
\end{aligned} \tag{7}$$

Suppose that the series $\sum_{k\geqslant 2} 2A_k x^{k-1}$ is convergent for $|x| < r$, $\Gamma_0 = [F'(x_0)]^{-1} \in \mathscr{L}(Y, X)$ exists for some $x_0 \in \Omega$. Assume the following Kantorovich conditions:

(C1) $\|\Gamma_0\| \leqslant \beta$,
(C2) $\|\Gamma_0 F(x_0)\| \leqslant \eta$,
(C3) $\|F''(x)\| \leqslant M,\ x \in \Omega$,
(C4) $\|F''(x) - F''(y)\| \leqslant K\|x - y\|,\ x, y \in \Omega,\ K > 0$.

We use the notation: $a_0 = M\beta\eta$, $b_0 = K\beta\eta^2$ and $v(x) = \sum_{k\geqslant 2} B_k x^{k-2}$, where $B_k = 2A_k$, $k\geqslant 2$.

From the initial conditions, we deduce $\|L_F(x_0)\| \leqslant a_0$, $K\|\Gamma_0\|\|\Gamma_0 F(x_0)\|^2 \leqslant b_0$. On the other hand, the existence of $\tilde{H}(L_F(x_0))$ is deduced if and only if $\|L_F(x_0)\| < r$. Then $a_0 < r$ is required.

From here, if $y_0 \in \Omega$, we obtain the following:

$$\|\tilde{H}(L_F(x_0))\| = \left\| I + \sum_{k\geqslant 2} 2A_k L_F(x_0)^{k-1} \right\| \leqslant 1 + \sum_{k\geqslant 2} B_k a_0^{k-1} = 1 + a_0 v(a_0),$$

$$\|x_1 - y_0\| = \|\tfrac{1}{2} L_F(x_0)\tilde{H}(L_F(x_0))\Gamma_0 F(x_0)\| \leqslant \tfrac{1}{2} a_0(1 + a_0 v(a_0))\|y_0 - x_0\|,$$

$$\|x_1 - x_0\| \leqslant \|x_1 - y_0\| + \|y_0 - x_0\| \leqslant (1 + \tfrac{1}{2} a_0(1 + a_0 v(a_0)))\eta.$$

Besides,

$$\|I - \tilde{H}(L_F(x_0))\| = \left\| \sum_{k\geqslant 2} 2A_k L_F(x_0)^{k-1} \right\| \leqslant \sum_{k\geqslant 2} B_k a_0^{k-1} = a_0 v(a_0).$$

Next, we try to generalize these initial bounds to any step of the iterative process. We then define the following real auxiliary sequences

$$a_{n+1} = a_n f(a_n)^2 g(a_n, b_n), \quad b_{n+1} = b_n f(a_n)^3 g(a_n, b_n)^2,$$

where

$$\begin{aligned}
&f(x) = \frac{2}{2 - 2x - x^2 - x^3 v(x)},\\
&g(x, y) = \frac{x^2}{2}\left[1 + (1 + x)v(x) + \frac{x}{4}(1 + xv(x))^2\right] + \frac{y}{6}.
\end{aligned} \tag{8}$$

We also consider the real auxiliary function $h(x) = (1 + \frac{1}{2}x(1 + xv(x)))$.

From these sequences and real auxiliary functions we prove by induction the following recurrence relations:

Lemma 1. *Let us suppose that* $y_0, y_n, x_n \in \Omega$, *for* $n \in \mathbb{N}$. *If* $a_0 < r$, $a_0 h(a_0) < 1$ *and the real auxiliary sequences* $\{a_n\}$ *and* $\{b_n\}$ *given by* (8) *are decreasing, then the following relations are verified*:

(I_n) $\Gamma_n = F'(x_n)^{-1}$ *exists and* $\|\Gamma_n\| \leqslant f(a_{n-1})\|\Gamma_{n-1}\|$,
(II_n) $\|\Gamma_n F(x_n)\| = \|y_n - x_n\| \leqslant f(a_{n-1})g(a_{n-1}, b_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|$,
(III_n) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \leqslant a_n$, $\|\tilde{H}(L_F(x_n))\| \leqslant 1 + a_n v(a_n)$,
(IV_n) $K\|\Gamma_n\|\|\Gamma_n F(x_n)\|^2 \leqslant b_n$,
(V_n)

$$\|x_{n+1} - x_n\| \leqslant h(a_n)\|\Gamma_n F(x_n)\|,$$

$$\|x_{n+1} - x_0\| \leqslant h(a_0)\left(\sum_{k=0}^{n} f(a_0)^k g(a_0, c_0)^k\right)\eta,$$

$$\|x_{n+1} - y_n\| \leqslant \tfrac{1}{2}a_n(1 + a_n v(a_n))\|\Gamma_n F(x_n)\|.$$

Proof. We begin proving that the conditions (I_n)–(V_n) are verified for $n = 1$. Firstly, we see that the inverse operator of $F'(x_1)$ exists and is bounded, since

$$F'(x_1) - F'(x_0) = \int_{x_0}^{x_1} F''(x)\,\mathrm{d}x = \int_0^1 F''(x_0 + t(x_1 - x_0))(x_1 - x_0)\,\mathrm{d}t,$$

it follows that $\|F'(x_1) - F'(x_0)\| \leqslant M\|x_1 - x_0\|$. Hence

$$\|I - \Gamma_0 F'(x_1)\| \leqslant (1 + \tfrac{1}{2}a_0(1 + a_0 v(a_0)))a_0 = h(a_0)a_0 < 1,$$

and, from Banach's Lemma [8], Γ_1 exists and (I_1) follows, since

$$\|\Gamma_1\| = \|\Gamma_1 F'(x_0)\Gamma_0\| \leqslant \|(\Gamma_0 F'(x_1))^{-1}\|\|\Gamma_0\| \leqslant \frac{\|\Gamma_0\|}{1 - h(a_0)a_0} = f(a_0)\|\Gamma_0\|.$$

Now, we prove (II_1). For this, we use the following integral decomposition for the operator F, obtained from Taylor's formula

$$F(x_{n+1}) = F(y_n) + F'(y_n)(x_{n+1} - y_n) + \int_{y_n}^{x_{n+1}} F''(x)(x_{n+1} - x)\,\mathrm{d}x, \tag{9}$$

$$F(y_n) = F(x_n) + F'(x_n)(y_n - x_n) + \int_{x_n}^{y_n} F''(x)(y_n - x)\,\mathrm{d}x. \tag{10}$$

On the other hand,

$$F'(y_n)(x_{n+1} - y_n) = \int_{x_n}^{y_n} F''(x)(x_{n+1} - y_n)\,\mathrm{d}x - \frac{1}{2}F''(x_n)\tilde{H}(L_F(x_n))(y_n - x_n)^2. \tag{11}$$

Then, if $x = x_n + t(y_n - x_n)$ in (10) and (11), and $x = y_n + t(x_{n+1} - y_n)$ in (9), we obtain

$$\begin{aligned} F(x_{n+1}) = & \int_0^1 (F''(x_n + t(y_n - x_n)) - F''(x_n))(y_n - x_n)^2(1-t)\,\mathrm{d}t \\ & + \int_0^1 F''(x_n)(I - \tilde{H}(L_F(x_n)))(y_n - x_n)^2(1-t)\,\mathrm{d}t \\ & + \int_0^1 F''(x_n + t(y_n - x_n))(y_n - x_n)(x_{n+1} - y_n)\,\mathrm{d}t \\ & + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(x_{n+1} - y_n)^2(1-t)\,\mathrm{d}t. \end{aligned}$$

Taking norms, for $n = 0$, it follows

$$\begin{aligned} \|F(x_1)\| \leqslant & \frac{K}{6}\|y_0 - x_0\|^3 + \frac{M}{2}a_0(1 + v(a_0) + a_0 v(a_0))\|y_0 - x_0\|^2 \\ & + \frac{M}{8}a_0^2(1 + a_0 v(a_0))^2\|y_0 - x_0\|^2. \end{aligned}$$

Thus,

$$\|\Gamma_1 F(x_1)\| \leqslant f(a_0)\left[\frac{b_0}{6} + \frac{a_0^2}{2}(1 + (1 + a_0)v(a_0)) + \frac{a_0^3}{8}(1 + a_0 v(a_0))^2\right]\|y_0 - x_0\|,$$

and we consequently obtain (II_1)

$$\|\Gamma_1 F(x_1)\| \leqslant f(a_0)g(a_0, b_0)\|\Gamma_0 F(x_0)\|.$$

Moreover,

$$\|L_F(x_1)\| \leqslant f(a_0)\|\Gamma_0\| M f(a_0) g(a_0, b_0)\|\Gamma_0 F(x_0)\| \leqslant a_0 f(a_0)^2 g(a_0, b_0) = a_1,$$

and, since $\{a_n\}$ is a decreasing sequence, it follows that $a_1 < r$. We now have that $\tilde{H}(L_F(x_1))$ exists,

$$\|\tilde{H}(L_F(x_1))\| = \left\| I + \sum_{k \geqslant 2} 2A_k L_F(x_1)^{k-1} \right\| \leqslant 1 + \sum_{k \geqslant 2} B_k a_1^{k-1} = 1 + a_1 v(a_1),$$

and (III_1) follows. As a consequence of the mentioned above, (IV_1) is immediate:

$$K\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 \leqslant K\beta\eta^2 f(a_0)^3 g(a_0, b_0)^2 = b_0 f(a_0)^3 g(a_0, b_0)^2 = b_1.$$

See that (V_1) is verified. Firstly,

$$\|x_2 - x_1\| = \|I + \tfrac{1}{2}L_F(x_1)\tilde{H}(L_F(x_1))\|\|\Gamma_1 F(x_1)\| \leqslant h(a_1)\|\Gamma_1 F(x_1)\|.$$

Besides,

$$\|x_2 - x_0\| \leqslant \|x_2 - x_1\| + \|x_1 - x_0\| \leqslant h(a_1)\|y_1 - x_1\| + h(a_0)\|y_0 - x_0\|,$$

and since $\{a_n\}$ is a decreasing sequence and the real function h is increasing, it follows that

$$\|x_2 - x_0\| \leqslant h(a_0)[1 + f(a_0)g(a_0, b_0)]\eta.$$

Furthermore,

$$\|I - \tilde{H}(L_F(x_1))\| = \left\| \sum_{k \geqslant 2} 2A_k L_F(x_1)^{k-1} \right\| \leqslant \sum_{k \geqslant 2} B_k a_1^{k-1} = a_1 v(a_1),$$

$$\|x_2 - y_1\| \leqslant \|\tfrac{1}{2} L_F(x_1)\tilde{H}(L_F(x_1))\| \|\Gamma_1 F(x_1)\| \leqslant \tfrac{1}{2} a_1(1 + a_1 v(a_1)) \|\Gamma_1 F(x_1)\|$$

and (V_1) is proved.

Now, by an induction process the result is easily verified. □

By means of these recurrence relations, we now prove the convergence of family of iterative processes (7).

4. Analysis of the real auxiliary sequences

Firstly, we give technical lemmas for the real functions f and g, as well as the auxiliary real sequences $\{a_n\}$ and $\{b_n\}$, whose proofs are immediate.

Lemma 2. *Let $f(x)$, $g(x, y)$ and $v(x)$ be the real functions given in* (8). *Then,*

(i) *If $h(a_0)a_0 < 1$, then $f(x)$ is increasing and $f(x) > 1$ for $x \in (0, a_0)$.*
(ii) *Fixed x, $g(x, y)$ is increasing as function of y. Besides, fixed y, $g(x, y)$ is increasing as function of x.*

Lemma 3. *If $b_0 < \kappa_0$, where $\kappa_0 = 12 + 6a_0 - 6h(a_0)(1 + 2a_0) + 3h(a_0)^2 a_0(2a_0 - 1)$, the sequences $\{a_n\}$ and $\{b_n\}$, given by* (8), *are decreasing. Besides, $f(a_0)g(a_0, b_0) < 1$.*

5. Main result of semilocal convergence

Theorem 4. *Let F be a nonlinear twice Fréchet-differentiable operator under the previous conditions. We suppose that $\Gamma_0 = [F'(x_0)]^{-1} \in \mathscr{L}(Y, X)$ exists, for some $x_0 \in \Omega$, and (I_n)–(V_n) are satisfied. We also assume that $a_0 < r$, $a_0 h(a_0) < 1$ and $b_0 < \kappa_0$. Then, if $B(x_0, R) \subset \Omega$, where*

$$R = \frac{h(a_0)\eta}{1 - f(a_0)g(a_0, b_0)},$$

the family of iterative processes given by (7) *starting in x_0, converges to a solution x^* of Eq.* (2). *In this case, the solution x^* and the iterations x_n belong to $\overline{B(x_0, R)}$ and x^* is a unique solution of* (2) *in $B(x_0, \frac{2}{M\beta} - R) \cap \Omega$.*

Proof. From recurrence relations obtained in Section 3 we have

$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta < R \quad \text{and} \quad \|x_1 - x_0\| \leqslant h(a_0)\eta < R,$$

and it follows that $y_0, x_1 \in \Omega$. We suppose that $y_k, x_{k+1} \in \Omega$, $k = 1, \ldots, n-1$, and we prove by induction that $y_n, x_{n+1} \in \Omega$. From recurrence relations (II$_n$) and (V$_n$), we have

$$\begin{aligned}\|y_n - x_0\| &\leqslant \|y_n - x_n\| + \|x_n - x_0\| \\ &< h(a_0)\left(\sum_{k=0}^{n} f(a_0)^k g(a_0, b_0)^k\right)\eta < R,\end{aligned}$$

$$\|x_{n+1} - x_0\| \leqslant h(a_0)\left(\sum_{k=0}^{n} f(a_0)^k g(a_0, b_0)^k\right)\eta < R.$$

Therefore $y_n, x_{n+1} \in \Omega$.

Besides, we have from recurrence relations (II$_n$) and (V$_n$),

$$\|x_{n+1} - x_n\| \leqslant h(a_n)\|\Gamma_n F(x_n)\| \leqslant h(a_0)\prod_{k=0}^{n-1} f(a_k)g(a_k, b_k)\eta.$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence and therefore convergent. Taking into account that $\{a_n\}$ and $\{b_n\}$ are decreasing sequences and $f(a_0)g(a_0, b_0) < 1$ it follows that

$$\begin{aligned}&\|x_{n+m} - x_n\| \\ &\quad\leqslant \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\quad\leqslant h(a_0)\left(\prod_{k=0}^{n+m-2} f(a_k)g(a_k, b_k) + \prod_{k=0}^{n+m-3} f(a_k)g(a_k, b_k) + \cdots + \prod_{k=0}^{n-1} f(a_k)g(a_k, b_k)\right)\eta \\ &\quad\leqslant h(a_0)\frac{1 - (f(a_0)g(a_0, b_0))^m}{1 - f(a_0)g(a_0, b_0)}\eta(f(a_0)g(a_0, b_0))^n.\end{aligned}$$

Thus, the sequence $\{x_n\}$ converges to a solution x^* of (2). We have

$$\|F(x_n)\| \leqslant \|F'(x_n)\Gamma_n F(x_n)\| \leqslant \|F'(x_n)\|\|\Gamma_n F(x_n)\| \leqslant \|F'(x_n)\|\prod_{k=0}^{n-1} f(a_k)g(a_k, b_k)\eta,$$

and since $\|F'(x_n)\|$ is bounded, then $\lim_{n\to\infty}\|F(x_n)\| = 0$. As a consequence of F is twice Fréchet-differentiable operator, it follows by continuity that $F(x^*) = 0$, and therefore x^* is a solution of (2).

Next, we consider y^* such that $F(y^*) = 0$, a solution of operator F with $y^* \in B(x_0, \frac{2}{M\beta} - R) \cap \Omega$. We suppose that y^* is a different solution of x^*. Then, it follows that

$$0 = \Gamma_0 F(y^*) - \Gamma_0 F(x^*) = \int_0^1 \Gamma_0 F'(x^* + t(y^* - x^*))\,\mathrm{d}t\,(y^* - x^*).$$

If the operator A^{-1} exists, where $A = \int_0^1 \Gamma_0 F'(x^* + t(y^* - x^*))\,\mathrm{d}t$, we get, $y^* - x^* = 0$ and then unicity of solution in $B(x_0, \frac{2}{M\beta} - R) \cap \Omega$ is obtained.

Then, from

$$
\begin{aligned}
&F'(x^*+t(y^*-x^*))-F'(x_0)\\
&\quad=\int_{x_0}^{x^*+t(y^*-x^*)} F''(x)\,\mathrm{d}x\\
&\quad=\int_0^1 F''(x_0+\tau(x^*+t(y^*-x^*)-x_0))[(x^*-x_0)(1-t)+t(y^*-x_0)]\,\mathrm{d}\tau,
\end{aligned}
$$

and taking norms, we get

$$\|F'(x^*+t(y^*-x^*))-F'(x_0)\|\leqslant M[\|x^*-x_0\|(1-t)+t\|y^*-x_0\|].$$

From $y^*\in B(x_0,\frac{2}{M\beta}-R)\cap\Omega$ and $x^*\in\overline{B(x_0,R)}\cap\Omega$, we have $\|y^*-x_0\|<\frac{2}{M\beta}-R$ and $\|x^*-x_0\|\leqslant R$. Therefore,

$$\|I-A\|\leqslant\|\Gamma_0\|M\int_0^1\|x^*-x_0\|(1-t)+t\|y^*-x_0\|\,\mathrm{d}t<1,$$

and, by Banach's lemma, the inverse operator of A exists and then $y^*=x^*$. $\square$

5.1. Particular cases

As particular cases of the Semilocal Convergence Theorem 4, given for family of iterative processes (7), we have the following results for the most known iterative methods:

1. *Chebyshev's method*: since $v(x)=0$, $h(x)=1+\frac{x}{2}$ and $r=+\infty$, Theorem 4 is satisfied if $a_0<\frac{1}{2}$ and

$$b_0<\tfrac{3}{4}(2+a_0)(2a_0-1)(-4+2a_0+a_0^2). \tag{12}$$

2. *The super-Halley method*: since $v(x)=\sum_{k\geqslant 2}x^{k-2}$, $h(x)=(x-2)/[2(x-1)]$ and $r=1$, Theorem 4 is verified if $a_0<0.380778$ and

$$b_0<\frac{3(8-32a_0+32a_0^2-9a_0^3+2a_0^4)}{4(a_0-1)^2}. \tag{13}$$

3. *Halley's method*: since $v(x)=\sum_{k\geqslant 2}1/2^{k-1}(x^{k-2})$, $h(x)=-2/(x-2)$ and $r=2$, Theorem 4 is verified if $a_0<0.434624$ and

$$b_0<\frac{6(4-12a_0+6a_0^2+a_0^3)}{(a_0-2)^2}. \tag{14}$$

Next, we represent the regions of cubic decreasing (see [2]) in Figs. 1 and 2, where a_0 and b_0 move in the cartesian axes. In Fig. 1, the discontinuous line represents the curve

$$b_0=\frac{6(1-2a_0)(4-2a_0-a_0^2)}{(2+a_0)^2},$$

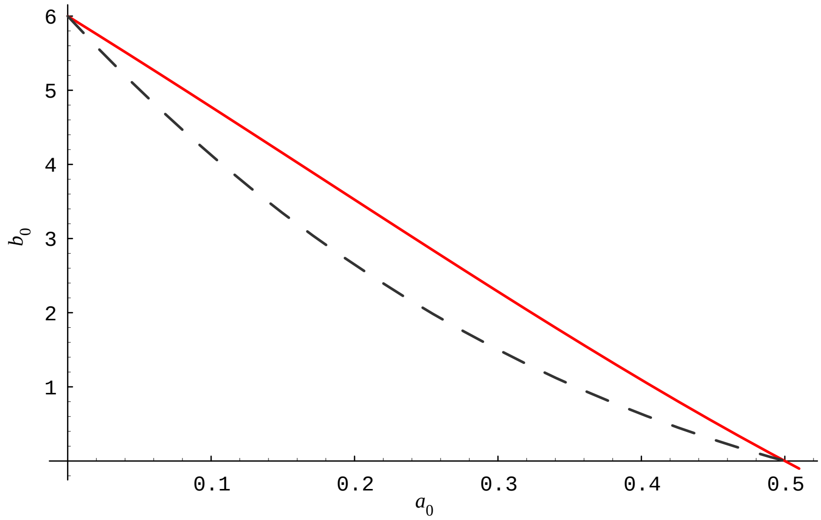

Fig. 1. Cubic decreasing regions for Chebyshev's method.

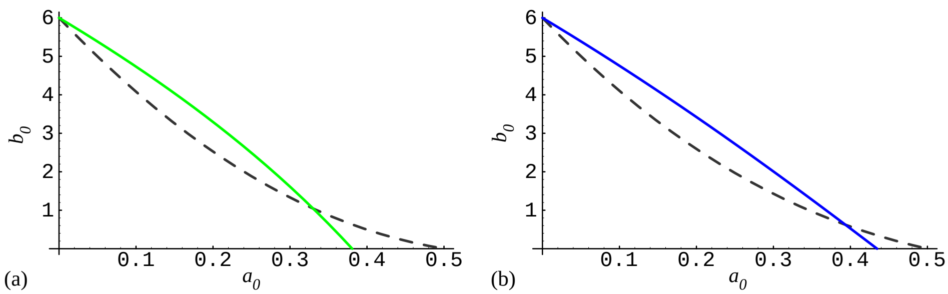

Fig. 2. Cubic decreasing regions for the super Halley and Halley methods.

which defines the region of cubic decreasing for Chebyshev's method obtained by Candela and Marquina in [2]. The continuous line represents curve (12) and defines the region of cubic decreasing given by Semilocal Convergence Theorem 4.

As consequence, the region of cubic decreasing defined by Theorem 4 is larger and we have therefore increased the region of accessibility for Chebyshev's method.

In Fig. 2, we compare the regions of cubic decreasing for super Halley and Halley methods with the ones given by Theorem 4. The discontinuous lines represent curves that define these regions

$$b_0 = \frac{6(1-2a_0)(1-a_0)(4-6a_0+a_0^2)}{(2-a_0)^2}$$

obtained in [6] for the super Halley method, and

$$b_0 = 3(1-a_0)(2-a_0)(1-2a_0)$$

obtained by Candela and Marquina in [2] for Halley's method.

The continuous lines represent the curves that define the regions of accessibility given in (13) and (14).

As it is observed in Fig. 2, when we compare the regions of accessibility obtained by Theorem 4 and the ones obtained in earlier results, we have that these regions are improved if $a_0 < 0.330098$ in the case of the super-Halley method and if $a_0 < 0.393553$ in the case of Halley's method.

Remark 1. The fact that we improve the regions of cubic decreasing in the case of Chebyshev's method and only for certain values of a_0 in the cases of the super-Halley and Halley methods, is due to the study presented here generalizes all the methods of R-order at least three, so that it is possible to optimize in occasions the bounds for the values a_0 and b_0 in the particular cases.

4. *Ostrowski's method*: since $v(x) = \sum_{k\geqslant 2} 2\binom{\frac{-1}{2}}{k} x^{k-2}$, $h(x) = x + (1/\sqrt{1+x})$ and $r = 1$, Theorem 4 is verified if

$$a_0 < 0.441439 \quad \text{and} \quad b_0 < 3\left(4-4a_0^2-a_0^3+2a_0^4-\frac{a_0(2a_0-1)}{1+a_0}+\frac{4a_0^3-2a_0^2-4a_0-2}{\sqrt{1+a_0}}\right).$$

5. *Euler's method*: since $v(x) = \sum_{k\geqslant 2}(-1)^k 2^{k+2}\binom{\frac{1}{2}}{k+1} x^{k-2}$, $h(x) = (1-\sqrt{1-2x})/x$ and $r = \frac{1}{2}$, Theorem 4 is verified if

$$a_0 < 0.365635 \quad \text{and} \quad b_0 < \frac{6(2-2\sqrt{1-2a_0}-3a_0+a_0^2)}{a_0}.$$

Remark 2. In the study of the most well-known methods of R-order at least three, we have established some bounds for a_0, that they are seemingly more restrictive that the demanded ones in Semilocal Convergence Theorem 4. This is a consequence of the fact that the value b_0 has to be positive.

5.2. R-order of convergence

Once we have studied the semilocal convergence of the multiparametric family of iterative processes given in (7), we see that it converges to a solution x^* of Eq. (2) with R-order of convergence at least three.

Lemma 5. *Let $\gamma = a_1/a_0$. From the hypotheses of Lemma 3, we have*

(i) $\gamma = f(a_0)^2 g(a_0, b_0) \in (0, 1)$,
(ii) $f(\gamma x) < f(x)$, $g(\gamma x, \gamma^2 y) < \gamma^2 g(x, y)$, $\forall x, y > 0$ *and* $\forall \gamma \in (0, 1)$.
(iii) $a_n \leqslant \gamma^{3^{n-1}} a_{n-1} \leqslant \gamma^{\frac{3^n-1}{2}} a_0$, $b_n \leqslant (\gamma^{3^{n-1}})^2 b_{n-1} \leqslant \gamma^{3^n-1} b_0$, $\forall n \geqslant 1$,
(iv) $f(a_n) g(a_n, b_n) \leqslant \gamma^{3^n} \Delta$, *where* $\Delta = \frac{1}{f(a_0)}$, $\forall n \geqslant 1$.

Proof. The proof of (i) and (ii) is immediate. To prove (iii), we use induction. If $n = 1$, $a_1 = \gamma a_0$. On the other hand, $b_1 \leqslant \gamma^2 b_0$, since

$$b_1 = b_0 f(a_0)^3 g(a_0, b_0)^2 < b_0 (f(a_0)^2 g(a_0, b_0))^2 = b_0 \gamma^2,$$

and $f(a_0)>1$. From Lemma 2 and (ii), it follows

$$a_{n+1}\leqslant\gamma^{3^{n-1}}a_{n-1}f(a_{n-1})^2(\gamma^{3^{n-1}})^2g(a_{n-1},b_{n-1})=\gamma^{3^n}a_n.$$

Hence

$$a_{n+1}\leqslant\gamma^{3^n}a_n\leqslant\gamma^{3^n}\gamma^{3^{n-1}}a_{n-1}\leqslant\cdots\leqslant\gamma^{\frac{3^{n+1}-1}{2}}a_0.$$

On the other hand,

$$b_{n+1}\leqslant(\gamma^{3^{n-1}})^2b_{n-1}f(a_{n-1})^3(\gamma^{3^{n-1}})^{2^2}g(a_{n-1},b_{n-1})^2=(\gamma^{3^n})^2b_n,$$

and then

$$b_{n+1}\leqslant(\gamma^{3^n})^2b_n\leqslant(\gamma^{3^n})^2(\gamma^{3^{n-1}})^2b_{n-1}\leqslant\cdots\leqslant\gamma^{3^{n+1}-1}b_0.$$

To conclude, we prove (iv). Taking into account (iii), we have

$$f(a_n)g(a_n,b_n)\leqslant f(a_0)\gamma^{3^n-1}g(a_0,b_0)\leqslant\gamma^{3^n}\frac{f(a_0)g(a_0,b_0)}{\gamma}=\frac{\gamma^{3^n}}{f(a_0)}=\gamma^{3^n}\Delta. \qquad \square$$

Remark. As a consequence of the three previous lemmas, it follows

$$\prod_{k=0}^{n-1}f(a_k)g(a_k,b_k)\leqslant\prod_{k=0}^{n-1}\gamma^{3^k}\Delta=\gamma^{\frac{3^n-1}{2}}\Delta^n.$$

Besides, since $\Delta<1$ and $\gamma<1$, it is clear $\lim_{n\to\infty}\prod_{k=0}^{n-1}f(a_k)g(a_k,b_k)=0$. Next, we see that family of iterative processes (7) has R-order of convergence at least three. For this, we remember the definition of R-order of convergence [10].

Definition 2. Let $\{x_n\}_{n\geqslant 0}$ be a sequence in a Banach space $(X,\|\cdot\|)$ with limit x^*. We say that the sequence $\{x_n\}_{n\geqslant 0}$ has R-order of convergence at least three if two constants $\delta\in(0,1)$ and $C\in\mathbb{R}^+$ exist such that

$$\|x_n-x^*\|\leqslant C\delta^{3^n}. \tag{15}$$

Theorem 6. *In the hypotheses of Theorem* 4, *the family of iterative processes given for* (7), *starting at* x_0, *converges to a solution* x^* *of Eq.* (2) *with R-order of convergence at least three. The following a priori estimates of the error are also obtained*

$$\|x^*-x_n\|<h\left(a_0\gamma^{\frac{3^n-1}{2}}\right)\eta\frac{\gamma^{\frac{3^n-1}{2}}\Delta^n}{1-\gamma^{3^n}\Delta}<(\gamma^{\frac{1}{2}})^{3^n}\frac{R}{\gamma^{\frac{1}{2}}}.$$

Proof. As a consequence of Lemma 5 and recurrence relations (II_n) and (V_n), we have the following:

$$\|x_{n+1}-x_n\|\leqslant h\left(a_0\gamma^{\frac{3^n-1}{2}}\right)\gamma^{\frac{3^n-1}{2}}\Delta^n\eta.$$

By using now the previous note, we obtain

$$\|x_{n+m} - x_n\| \leqslant \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\|$$
$$\leqslant h\left(a_0\gamma^{\frac{3^n-1}{2}}\right)\left(\gamma^{\frac{3^{n+m-1}-1}{2}}\Delta^{n+m-1} + \gamma^{\frac{3^{n+m-2}-1}{2}}\Delta^{n+m-2} + \cdots + \gamma^{\frac{3^n-1}{2}}\Delta^n\right)\eta.$$

By the Bernouilli inequality, $(1+x)^k > 1 + kx$, we have that $3^k - 1 > 2k$, and therefore

$$\gamma^{\frac{3^n-1}{2}}\Delta^n\left(\gamma^{\frac{3^n(3^{m-1}-1)}{2}}\Delta^{m-1} + \gamma^{\frac{3^n(3^{m-2}-1)}{2}}\Delta^{m-2} + \cdots + 1\right) < \gamma^{\frac{3^n-1}{2}}\Delta^n\frac{1-(\gamma^{3^n}\Delta)^m}{1-\gamma^{3^n}\Delta}.$$

Then, to apply the Bernouilli inequality, we have for $n, m \in \mathbb{N}$,

$$\|x_{n+m} - x_n\| < h\left(a_0\gamma^{\frac{3^n-1}{2}}\right)\gamma^{\frac{3^n-1}{2}}\Delta^n\eta\frac{1-(\gamma^{3^n}\Delta)^m}{1-\gamma^{3^n}\Delta}. \tag{16}$$

Then, if $m \to \infty$ in (16), we obtain:

$$\|x^* - x_n\| < h\left(a_0\gamma^{\frac{3^n-1}{2}}\right)\eta\frac{\gamma^{\frac{3^n-1}{2}}\Delta^n}{1-\gamma^{3^n}\Delta} < (\gamma^{\frac{1}{2}})^{3^n}\frac{R}{\gamma^{\frac{1}{2}}}.$$

Hence it is proved that the family of iterative processes has R-order of convergence at least three, since (15) with $C = R/\gamma^{1/2}$ and $\delta = \gamma^{1/2} < 1$. □

Remark. As a consequence of the previous theorem it is proved that the iterative processes of Chebyshev, super-Halley, Halley, Ostrowski, and Euler, have R-order of convergence at least three.

5.3. Numerical tests

Next, we give two examples that illustrate the previous results. We give a priori error bounds for the Chebyshev method and those α-methods given by

$$x_{n+1} = x_n - (I + \tfrac{1}{2}L_F(x_n) + \alpha L_F(x_n)^2)\Gamma_n F(x_n), \tag{17}$$

improving the bounds obtained previously.

Example 1. We consider the operator F given by $F(x) = x^3 - 10$, $x_0 = 2$ and we denote the positive root of $F(x) = 0$ by x^*. We give an upper bound C of $10^{11}\|x^* - x_2\|$, where x_2 is the second iteration of the Chebyshev method. Considering the domain $(1, 3)$, we have that $\beta = \frac{1}{12}$, $\eta = \frac{1}{6}$, $M = 18$ and $K = 6$. Consequently $a_0 = \frac{1}{4}$ and $b_0 = \frac{1}{72}$. Making the same decomposition that Candela and Marquina in [2] and calculating the smaller value of n so that $\|x^* - x_n\|$ is of order 10^{-11}, we consider

$$\|x^* - x_2\| \leqslant \|x^* - x_n\| + \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_3 - x_2\|.$$

In our case, $n = 3$ and we obtain $C = 130931.5282$. For the same operator and iterative method, Candela and Marquina obtain $C = 142360.973$ in [2].

Table 1
Estimates for $\alpha = \frac{1}{2}$

n	$\|x^* - x_n\|$	$\|x^* - x_n\|$, ($\alpha = \frac{1}{2}$) in [4]
0	0.28709603070577	0.2870424876072915
1	0.00062292490294	0.0007359920056491
2	$2.28337 \cdot 10^{-14}$	$4.48122 \cdot 10^{-14}$
3	$6.75869 \cdot 10^{-56}$	$6.16436 \cdot 10^{-55}$

Example 2. We consider the space $X = C([0, 1])$ of continuous functions defined in the interval $[0, 1]$ with the norm $\|x\| = \max_{s\in[0,1]} |x(s)|$. We consider the equation $F(x) = 0$, where

$$F(x)(s) = 1 - x(s) + \frac{x(s)}{4} \int_0^1 \frac{s}{s+t} x(t)\, dt, \quad x \in C([0, 1]), \; s \in [0, 1].$$

This is the quadratic equation of Chandrasekhar [1]. With the previous notation and $x_0 = x_0(s) = 1$, we calculate the first and second Fréchet derivative of F to obtain the following constant $\beta = 1.5303942\ldots$, $\eta = 0.2651971\ldots$ and $M = 0.34657359\ldots$. Therefore, the Chandrasekhar equation has a solution in $\overline{B(x_0, R)}$, with $R = 0.28709603\ldots$ and is unique in $B(x_0, \frac{2}{M\beta} - R) \cap \Omega$, with $\frac{2}{M\beta} - R = 3.4836841\ldots$. In Table 1, we show some estimates of the error $\|x^* - x_n\|$, for the method given in (17), with $\alpha = \frac{1}{2}$, that was considered in [4]. We see that the estimates of the error improve the estimates obtained there.

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